# Sommario — Abstract

*E' opinione corrente che la geometria noncommutativa possa costituire il sistema di concetti appropriato alla descrizione dello spaziotempo alla piccolissima scala (la lunghezza di Planck* $\lambda_P$*). Il paradigma della geometria noncommutativa consiste nel concepire gli spazi localmente compatti come casi particolari del concetto più generale di C*-algebra, al quale sono equivalenti nel caso abeliano. Ad una C*-algebra non commutativa di "funzioni", non corrisponde alcuno spazio in senso stretto—ossia inteso come insieme di punti—tuttavia è possibile accedere alla geometria soggiacente valutando gli stati sulle "funzioni" (gli stati sono la generalizzazione al caso noncommutativo delle misure normalizzate sulla C*-algebra* $\mathcal{C}_0(X)$ *delle funzioni sullo spazio* $X$*). Di conseguenza, nel caso noncommutativo i punti privi di estensione (che classicamente corrispondono alle misure di Dirac) cessano di esistere, lasciando luogo ad un concetto più sfumato ("fuzzy").*

*Una classe di modelli di spaziotempo noncommutativo, basati sulla quantizzazione delle coordinate, è stata introdotta per la prima volta in [12, 13], dove si considerava un sistema di relazioni di indeterminazione delle coordinate, fisicamente motivate da una condizione di stabilità dello spaziotempo, quando sottoposto a misure della posizione. Fu possibile reperire un sistema di relazioni di commutazione delle coordinate, tali che le corrispondenti relazioni di indeterminazione di Heisemberg fossero compatibili con le relazioni di indeterminazioni date. Infine, in quegli stessi articoli veniva completamente risolto il problema della teoria delle rappresentazioni (regolari, covarianti) di tali relazioni di commutazione tramite l'introduzione di operatori autoaggiunti* $q_\mu$*, associati alle rappresentazioni di una C*-algebra universale* $\mathcal{E}$*. Il limite classico di* $\mathcal{E}$*, al tendere di* $\lambda_P$ *a zero, descrive il prodotto cartesiano* $\mathbb{R}^4 \times \Sigma$*, dove* $\Sigma$ *è lo spettro del centro* $Z$ *di* $M(\mathcal{E})$*. Gli stati di ottima localizzazione (compatibilmente cone le relazioni di incertezza) sono stati classificati e sono stati fatti i primi passi verso la teoria dei campi quantistici su tale modello.*

*Nel presente lavoro, proseguiamo il programma delineato poc'anzi. Si propone e si motiva la costruzione della "algebra delle funzioni di n punti" come il prodotto tensoriale di Z-moduli* $\mathcal{E}^{(n)} = \mathcal{E} \otimes_Z \cdots \otimes \mathcal{E}$*, nonché se ne descrive la dualità in termini di operatori autoaggiunti illimitati* $q_\mu^j$ *che soddisfino le relazioni di commutazione per ogni j fissato, e che commutino per valori distinti*

*di j. Si descrive quindi una soluzione al problema di definire una ragionevole generalizzazione noncommutativa della valutazione su punti coincidenti di funzioni classiche. La costruzione è resa possibile dal fatto che le coordinate del centro di massa* $\bar{q}_\mu = (q_\mu^1 + \ldots + q_\mu^n)/n$ *sono statisticamente indipendenti dalle separazioni* $q_\mu^i - q_\mu^j$ *per ogni coppia* $(i, j)$*, rendendo così possibile la definizione di una traccia parziale che minimizzi le separazioni.*

*Tali risultati permettono di dare una nuova definizione di potenza in ordine normale di operatori di campo come prodotti di campi a "punti" diversi, successivamente valutati "a punti coincidenti", di modo da non trascurare la noncommutatività, e inoltre ottenendo un effetto regolarizzante.*

$\star$

*Noncommutative geometry is currently believed to provide a suitable framework for the description of the Space Time at the smallest scale (the Planck length* $\lambda_P$*). The paradigm of noncommutative geometry is to conceive locally compact spaces as particular cases of the more general concept of C*-algebra, to which they are equivalent in the abelian case. In the case of a noncommutative C*-algebra of "functions", strictly speaking there is no corresponding space (intended as a set of points), yet it is possible to access the underlying geometry by evaluating the states on the "functions" (the states are the noncommutative generalization of the normalized measures on the commutative C*algebra* $\mathcal{C}_0(X)$ *of the functions of the space* $X$*). As a consequence, in the noncommutative case extensionless points (which classically correspond to the Dirac measures) cease to exist, and leave place to a more "fuzzy" concept.*

*A class of models of noncommutative spacetime, based on the quantization of the coordinates, has been pioneered in [12, 13] starting from a set of uncertainty relations on the coordinates, physically motivated by a condition of stability of the spacetime* *under localization. It was possible to find a set of commutation relations among the coordinates, such that the corresponding Heisenberg uncertainty relations were in agreement with the given uncertainty relations. Finally, the representation problem of these relations (integrated in the Weyl form, and fulfilling natural Poincaré covariance) was solved in terms of a covariant set of unbounded, selfadjoint operators* $q_\mu$*, associated to the representations of a universal* $C^*$*-algebra* $\mathcal{E}$*. The classical limit of* $\mathcal{E}$ *as the Planck length* $\lambda_P$ *goes to zero describes the cartesian product* $\mathbb{R}^4 \times \Sigma$*, where* $\Sigma$ *is the spectrum of the center* $Z$ *of* $M(\mathcal{E})$*. The states with optimal localization (compatibly with the uncertainty relations) were classified, and the first steps of quantum field theory on this model were also laid down.*

*In this research, we carry on the above program. We propose and motivate the construction of the algebra of "functions of n points" as the n-fold tensor product of* $Z$*-moduli* $\mathcal{E}^{(n)} = \mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E}$*, and describe its representation theory in terms of unbounded, selfadjoint operators* $q_\mu^j$*, fulfilling the basic relations for fixed* $j$*'s, and strongly commuting for different* $j$*'s. Next, we give a solution to the problem of finding a suitable replacement for the evaluation at coincident points of classical functions. The solution is made possible by the remark that the center of mass coordinates* $\bar{q}_\mu = (q_\mu^1 + \ldots + q_\mu^n)/n$ *are statistically independent from all the pairwise separations* $q_\mu^j - q_\mu^k$*, and allow to define a partial trace by minimizing the pairwise separations.*

*The above results allow for a definition of the normal ordering of products of field operators evaluated "at coincident points" (Wick powers), which takes into account the noncommutativity of the Space Time, and has a regularizing effect.*

# Chapter 1

# Introduction

## 1.1 Non commutative algebras, and geometries without points.

*The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.*

G. F. B. Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen", 1854 (English Translation by M. Spivak)

Riemann was probably the first who realized that the extension of the Continuum hypothesis to the infinitely small scale was not an idle question. About seventy years later, quantum mechanics came into the play; operators on the Hilbert space were called "q-numbers" by the physicists, to stress their being a sort of generalization of the "c-numbers", the "classical" numbers. Possibly discrete spectra arose as a consequence of the non commutativity of the product of operators, and were to represent the possible results of the measurements of physical quantities. Nevertheless, quantization only concerned the description of the physical systems: the continuity of the ambient space was not questioned. The fundamental papers of von Neumann on operator theory and operator algebras opened the way to a beautiful and new branch of mathematics, the theory of C*-algebras, which provides a natural framework for algebraic methods in functional and abstract harmonic analysis, as well as a natural mathematical language for many of the most subtle issues of quantum physics.

It was soon realized that C*-algebras were providing a natural non commutative generalization of some classical notions in topology and measure theory. Actually, the abstract algebraic structure of some abelian C*-algebra $\mathcal{A}$ (with unit) is sufficient to fully characterize a locally compact (resp. compact) space $X$, unique up to isomorphisms, such that $\mathcal{A}$ is isomorphic to the C*-algebra $\mathcal{C}_0(X)$ of the continuous functions of $X$, vanishing at infinity. Hence, general C*-algebras appeared to be natural candidates for the algebras of "functions" of possibly non commutative topological spaces. The theory of traces and weights on operator algebras also appeared to share many features with ordinary measure theory, to which it reduces in the abelian case ([29], see also [33]). Starting from the eighties, Alain Connes, who had already given fundamental contributions to the theory of operator algebras, gave new impetus to the whole subject of noncommutative geometry. In particular, his theory of the spectral triples provides a natural noncommutative generalization of the theory of compact, riemannian manifolds. The interested reader is referred to its beautiful book [5], and to the recent review [6].

It is remarkable that non commutative spaces are not made of points: it is not possible to derive from a noncommutative algebra of "functions" a set of objects to be called "points" in any reasonable sense; we only can access the non commutative space indirectly through its "functions", and the corresponding positive functionals on them (see also the end of section 1.2). In a sense, this suggests that points are a derived concept, which may or may not be encompassed by a geometry: points do appear precisely when the geometry is "commutative", i.e. classical, in which case the functional $f \mapsto f(x)$ is positive for each fixed $x$. This point of view is supported by the remark that some uninteresting pathological measure spaces, which can be constructed by means of some very peculiar and exotic sets of points, are easily ruled out in the reformulation of ordinary measure theory in the language of weights on operator algebras.

## 1.2 A model for the flat quantum space time

*Si possono dare delle buone ragioni del perché la realtá non possa essere rappresentata con un campo continuo. Dai fenomeni*

> *quantistici sembra discendere con certezza che un sistema finito di energia possa essere compiutamente descritto da un insieme finito di numeri (numeri quantici). Questo non sembra in accordo con una teoria del continuo, e deve condurre al tentativo di trovare una teoria puramente algebrica per la descrizione della realtà. Tuttavia nessuno sa in che modo ottenere le basi di una tale teoria.*
>
> A. Einstein, "The Meaning of Relativity", 1952

Noncommutative geometry is expected to provide a natural framework for the investigation of the structure of the physical spacetime in the small. Of course the relevant issue is to detect some physical principles which dictate the algebraic relations between the "functions" of a physically reasonable model of spacetime.

It is a long standing conjecture, that strict Einstein locality should break down in the "small". Actually "small" is thought to be of order of the Planck length

$$\lambda_P = \sqrt{\frac{G\hbar}{c^3}} \simeq 1.6 \times 10^{-33} \text{ cm},$$

which is the unique length which can be obtained by combining the gravitational constant $G$, the rationalized Planck constant $\hbar$, and the light speed $c$. In other words, one expects that the breakdown of strict locality occurs in the regime of high energy quantum physics, and is related to gravitation: in particular, $\lambda_P$ equals both the Compton wavelength and (twice) the Schwarzschild radius of the gravitational field of a point particle of mass $M_P = \hbar/(c\lambda_P)$. In this connection, see [27], and references therein. These remarks provided the starting point for several investigations from quite different points of view (see the references [1-6] of [13], and the more recent, but not comprehensive, bibliography of [14]).

In the framework of noncommutative geometry à la Connes, only models of compact "euclidean" (i.e. riemannian) noncommutative spaces are available. This is somehow disturbing, as far as high energy physics is concerned, since it is widely believed that the physical spacetime is pseudoriemannian with lorentzian signature (and possibly non compact). The euclidean formulation is common practice in high energy physics (on the classical spacetime), since it is possible—under quite general assumptions on the mathematical nature of the operator fields $\phi$—to perform Wick "rotations", i.e. suitable analytic continuations of the Wightman functions[1] from $M \times \cdots \times M$ (here

[1] The $n^{\text{th}}$ Wightman "function"—actually a distribution in $\mathcal{S}(\mathbb{R}^{4(n-1)})')$—associated to

$M$ is the Minkowski spacetime, which we may identify with $\mathbb{R}^4$) to certain open regions in $M_c \times \cdots \times M_c$, where $M_c = M + iM = \mathbb{C}^4$. Unfortunately, there is not—as of today—any reasonable noncommutative generalization of the commutative concept of analyticity, or any other equivalent notion of "rigidity", nor even any idea of what a noncommutative complexification of a noncommutative space ought to be. The fact is, that we simply do not know how to relate noncommutative euclidean spaces to any noncommutative model of the physical (pseudoriemannian) spacetime. So far, then, the relevance to high energy physics of noncommutative euclidean spaces is far from being unquestionable.

A physically motivated class of noncommutative analogues of the flat Minkowski spacetime was first proposed in [12, 13], were the algebraic relations between the "functions" are constrained by operationally motivated Heisenberg–like uncertainty relations among the coordinates, which are postulated as a stability condition of the Space Time under localization in the small. This is much in the spirit of the beginnings of Quantum Mechanics, where the algebraic relation $[q, p] = \hbar I$ was introduced as a solution to the problem of finding operators fulfilling $\Delta p \Delta q \gtrsim \hbar/2$ in any state. See [12] for a less technical of the results in [13].

In the present research, we shall focus on a particular model in the class introduced in [13], which we shall call in the sequel the *DFR quantum spacetime.*

The approach of [13] was the following: first, the authors derived from a stability condition on the quantum spacetime in terms of classical gravity, a reasonable ansatz for a set of spacetime uncertainty relations between the uncertainty $\Delta x_\mu$ of the coordinates. Then they proposed a solution to the following problem: find four selfadjoint operators $q_0, q_1, q_2, q_3$ such that

1. their commutation relations (in the regular form) induce Heisenberg

the operator field $\phi$ is the vacuum expectation $W^{(n)}(\xi_1, \ldots, \xi_{n-1}) = (\Omega, \phi(x_1) \cdots \phi(x_n)\Omega)$, where $\xi_i = x_{i+1} - x_i$, and the definition is unambiguous by the invariance of the vacuum under translations. The Wick "rotation"of the $(n+1)^{\text{th}}$ Wightman function, then, is the analytic continuation–made possible by the spectrum condition—of $W^{(n+1)}$ to the "forward tube" in $\mathbb{C}^{4n}$. The restrictions $S^{(n)}(\eta_1, \ldots, \eta_n) = W^{(n+1)}(i\eta_1, \ldots, i\eta_n)$ to the euclidean region of such analytic continuations—also known as the Schwinger functions—are, by Österwalder-Schrader positivity, the $n$-points functions of some stochastic process on $\mathbb{R}^4$, via the natural identification of the euclidean region in $\mathbb{C}^{4n}$ with $\mathbb{R}^{1,3} \times \cdots \times \mathbb{R}^{1,3}$. The euclidean formulation is completely equivalent to the operator field formulation, since it is possible to reconstruct the operator fields out of the Wightman functions, by means of GNS–like techniques (see e.g [22]). This allows to conceal the difficulties inherent to the analysis of unbounded operators, and to use the powerful techniques of ordinary classical statistical mechanics and of the theory of stochastic processes, see [19].

uncertainty relations compatible with the spacetime uncertainty relations (writing $\Delta q_\mu = \Delta x_\mu$),

2. they are Poincaré covariant, and

3. their commutators should vanish in the large scale limit.

The uncertainty relations arising from the basic solution—called by the authors the *weak* spacetime uncertainty relations—do not saturate the basic relations. Moreover, the solution to this problem cannot be expected to be unique: another solution enforcing the weak commutation relations was proposed in [9] (see also [10] for an outlook). Nevertheless, there is an essentially unique solution in the case of the weak relations, if one requests in addition that the commutators $[q_\mu, q_\nu]$ are central. This leads precisely to the DFR quantum spacetime, which is a manageable one, though not entirely unrealistic.

The assumption that Poincaré covariance holds in the small is quite natural, since otherwise it would break down also at the macroscopic scale (see also section 2.2). Unfortunately, it is responsible for an unpleasant feature of the DFR quantum spacetime: the algebra of "functions" has a highly nontrivial center, which seems to suggest that the corresponding space–time is not fully quantized. Actually, the center can be essentially identified with the algebra $Z$ of bounded, continuous functions of some locally compact space $\Sigma$, which survives the classical limit as a cartesian factor of $\mathbb{R}^4$. The rôle of $Z$ is still to be understood, but see [10].

The spacetime uncertainty relations were operationally motivated. Since a measurement of the position $x_\mu$ within an accuracy $\Delta x_\mu$ involves an energy transfer of order $1/\Delta_\mu$, then the measurement of all the coordinates within the accuracies $\Delta x_\mu$ involve an energy transfer of order $1/a$, with $a = \min\{\Delta x_\mu : \mu = 0, \ldots, 3\}$. The resulting state is localized at some time in space within the accuracies $\Delta x_1, \Delta x_2, \Delta x_3$, and has an energy tensor $T_{\mu\nu}$ with total energy $1/a$ (in natural units). The stability condition on the $\Delta x_\mu$ arises from the request that $T_{\mu\nu}$ "should not be such to trap photons. For otherwise the concentration of energy needed for the localization experiment would have the catastrophic effect of giving rise to black hole formation and thus putting the events under study out of the reach of observation. A heuristic estimate leads to a very reasonable ansatz, namely the *weak* spacetime uncertainty relations

$$\Delta x_0 \sum_{j=1}^{3} \Delta x_j \gtrsim \lambda_P^2,$$

$$\sum_{i\leq j<k\leq 3} \Delta x_j \Delta x_k \gtrsim \lambda_P^2. \text{ ''}$$

This short account on the spacetime uncertainty relations closely patterns the one given in [12]; for more details, see [13].

The above mentioned solution with central commutators is the following:

$$Q_{\mu\nu}Q^{\mu\nu} = 0,$$

$$\left(\frac{1}{2}Q_{\mu\nu}(*Q)^{\mu\nu}\right)^2 = \lambda_P^8 I,$$

$$[q_\mu, Q_{\nu\rho}] = 0,$$

where $Q_{\mu\nu} = -i[q_\mu, q_\nu]$, and $*Q$ is its Hodge dual.

The representation theory of the universal algebra of the above relations was completely described in [13] (see also section 2.1). The generators $q_\mu$ turned out to have purely continuous spectrum coinciding with the real line $\mathbb{R}$. This was possible because the above commutation relations essentially define a finitely generated Lie algebra, which made it possible to extend the standard machinery of twisted convolution products used in von Neumann's proof of the uniqueness of the Weyl relations ([28], but see the more modern [23]). More precisely, one may derive a twisted convolution product, also referred to as the Weyl–Moyal $\star$-product[2], which is induced by the identifi-

[2]Actually, by this is usually meant—in the current literature— the configuration space product $\star$ arising by Fourier analysis from the twisted convolution product $\times$ in momentum space. The latter is defined, with $q$ an irreducible set of quantum coordinates fulfilling $[q_\mu, q_\nu] = i\sigma_{\mu\nu}I$, by $\int d^4k\ (\check{f} \times \check{g})(k)e^{ikq} = f(q)g(q)$, and reads

$$(\check{f} \times \check{g})(k) = \int d^4h\ \check{f}(h)\check{g}(k-h)e^{\frac{i}{2}\sigma^{\mu\nu}k_\nu h_\nu}.$$

By standard Fourier theory, one may define

$$(f \star g)(x) \equiv \widehat{(\check{f} \times \check{g})}(x) = \int d^4y d^4z\ f(y)f(z)C_2(y-x, z-x)$$

for a suitable singular kernel $C_2$ (see [13, Appendix C]). If $f$ and $g$ are $\mathcal{C}^\infty$ and the Taylor expansions of $f$, $g$, and $f \star g$ converge in a common neighborhood of $x$, then

$$(f \star g)(x) = e^{\frac{i}{2}\sigma^{\mu\nu}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\eta_\nu}} f(x+\xi)g(x+\eta)\bigg|_{\xi=\eta=0},$$

up to irrelevant Plancherel factors.

cation between the (represented) element

$$f(q)=\int_{\mathbb{R}^4} d^4k\ \check{f}(k)e^{ik^\mu q_\mu}$$

of (the multipliers algebra of) the algebra of "functions" on the DFR quantum spacetime, and its symbol $f \in \mathcal{S}(R^4)$, $f=f(x)$. The above identification also provides the interpretation of the classical limit (in this connection, see also [31]).

The irreducible representations $q^\sigma$ fulfill $[q^\sigma_\mu, q^\sigma_\nu]=i\sigma_{\mu\nu}I$ for $\sigma$ a matrix in suitable set $\Sigma$. Hence the full algebra $\mathcal{E}$ is essentially (i.e. up to the embedding in the corresponding universal C*-algebra) an algebra of continuous $L^1$–valued functions $\varphi=\varphi(\sigma,\cdot)$ of $\sigma$, endowed with the convolution product twisted by $\sigma$ at each fixed $\sigma$ (see section 2.1 for more details). Actually $\check{f}(k)$ is not in $\mathcal{E}$, but is in the multipliers algebra $M(\mathcal{E})$.

The DFR quantum spacetime is a geometry without points, since for any $x \in \mathbb{R}^4$, the map $f(q) \mapsto f(x)$ fails to be positive. The states on the algebra $\mathcal{E}$ of "functions" are then extended objects. A notion of states with optimal localization was introduced in [13], and is carefully described in section 2.1 (see also section 1.4 for a less technical description) ; however, so far any conceivable notion of localization seems to be a priori at best Galilei covariant, since a suitable Lorentz boost may delocalize any extended object. This fact is a source of many difficulties.

An interesting feature of the spacetime uncertainty relations is that they are compatible with exact localization in time, at the cost of total delocalization in space. For this reason, the map

$$f(q)\mapsto\int_{\mathbb{R}^3} d\mathbf{x}\ f(t,\mathbf{x})$$

is positive, which motivates the notation

$$\int_{q_0=t} d^3q f(q)\equiv\int_{\mathbb{R}^3} d\mathbf{x}\ f(t,\mathbf{x}).$$

Of course, also

$$\int d^4q\ f(q)\equiv\int dt\int_{q_0=t} d^3q\ f(q)=\int_{\mathbb{R}^4} d^4x\ f(x)$$

is positive.

It is worth to stress that the stability condition underlying the spacetime uncertainty relations is not meant to discard black holes formation at a scale bigger than that of the Planck length. The DFR quantum spacetime is to be thought of as a semiclassical one, in which the energies involved are high enough to bring into the play the noncommutativity of the Space Time, but too feeble to induce any non zero curvature.

We conclude this section with some historical remarks. To the best of the author's knowledge, quantization of the Space Time induced by operationally motivated Heisenberg uncertainty relations among the coordinates, was first proposed in [12, 13]. Quantization of the spacetime by means of noncommutative coordinates, however, already appeared in [32], with different motivations. The aim of Snyder was not to quantize the spacetime on the basis of physical principles. In the forties, before the inception of the renormalization program, physicists were struggling against the ultraviolet divergences in quantum field theory, trying to replace the space–time with some lattice of points separated by the length $1/M$, where $M$ was the rest mass of the heaviest particle in the theory. The drawback of this approach was of course the nearly complete breakdown of covariance. Snyder's idea was to introduce quantized coordinates with discrete spectra in order to mimic lattice regularization in a covariant way. The covariance of the coordinates was defined as the invariance of the spectrum of the coordinates under the linear transformations $\Lambda : t, x, y, z \mapsto t', x', y', z'$ preserving the form $t^2 - x^2 - y^2 - z^2$, which of course was a very weak notion of covariance, too weak, e.g., to ensure the existence of a unitary $U(\Lambda)$ inducing the above transformation of coordinates by adjoint action. Snyder found a solution to the above problem: his coordinates $x, y, z$ all had spectra of the form $\mathbb{N}a$, with $a = 1/M$ the Compton wavelength of the heaviest particle of the theory, while the time $t$ resulted continuous. Moreover, they were fully covariant in the restricted sense of above. Covariance under infinitesimal translations was broken anyway. The discrete spectra of the space coordinates were expected to produce the desired regularization on the quantum fields. No interpretation of the uncertainty relations resulting from the noncommutativity of the coordinates was provided[3]. Snyder's brilliant ideas were essentially abandoned when the renormalization program took off.

[3]In Snyder's own words: *We will not discuss in any detail in this paper the limitations placed upon the simultaneous measurability of $x$, $y$, $z$, and $t$ due to the noncommutativity of these quantities. Some preliminary calculations which I have made indicate that these limitations are not serious enough to interfere with the ordinary description of atomic phenomena in terms of a continuous space-time nor with our usual macroscopic theory.*

## 1.3 Quantum field theory on the quantum space time

*The crux of the interface of quantum physics and gravitation is, however, the short distance regime. There, below $10^{10}\lambda_P$ we have no direct guidance from experiment and cannot expect any. We can speculate and try to produce a scheme whose merits can be tested by establishing contact with extrapolations from high energy physics and ultimately by an understanding of the relation of mass scales in particle physics to the Planck length.*

R. Haag, "Local quantum Physics", 1996

In the absence of anything equivalent to the Wick rotation (see page 4), it seems that—so far—the only *direct* approach to quantum field theory on the quantum spacetime is in terms of operator fields. In [13], the very first steps were taken towards a theory of fields of operators on the quantum spacetime.

The map $f \mapsto f(q) = \int d^4k\ \check{f}(k)e^{ikq}$ from the continuous functions in $L^1(\mathbb{R}^4)$ to the (represented) elements of the multipliers algebra $M(\mathcal{E})$, can be easily extended to operator valued functions in $L^1(\mathbb{R}^4, \mathcal{B}(\mathfrak{H}))$. Quantum fields on the usual (i.e. classical) Minkowski spacetime, however, are not operator valued functions; they are maps from the test functions of $\mathbb{R}^4$ into the polynomial *-algebra of the closable, unbounded operators with some common core $D \subset \mathfrak{H}$, and they are distributions in the sense that, for each $\Phi, \Psi \in D$, $f \mapsto (\Phi, \phi_m(f)\Psi)$ is an ordinary distribution (see [22] for more details).

In the case of the free Bose field $\phi_m$ on the usual (classical) spacetime, the expression

$$\check{\phi}_m(f) = \int d^4\ k\check{\phi}_m(k)f(k)$$

of the Fourier transform $\check{\phi}(f) \equiv \phi(\check{f})$ of $\phi_m$ is not only formal : actually it holds as an equation between quadratic forms with form domain $D$ (see e.g. [30], or appendix D), and $\check{\phi}_m(k)$ itself is such a form. Hence

$$\phi_m(q) = \int d^4k\ \check{\phi}_m(k)e^{ikq},$$

is a well defined map from states on $\mathcal{E}$ to quadratic forms with form domain $D \otimes \mathfrak{H}_q$, where $\mathfrak{H}_q$ is the representation space of the covariant, nondegenerate coordinates $q$. Defining, for any state $\omega$ in $\mathcal{S}(\mathcal{E})$ and with $\tilde{\omega}$ its normal extension to $M(\mathcal{E})$, the function

$$\check{\psi}_\omega(k) = \tilde{\omega}(e^{ikq}),$$

we have

$$\langle \omega, \phi_m(q) \rangle = \phi_m(\psi_\omega),$$

where the l.h.s. is the usual Wightman free bosonic field evaluated on the ordinary test function $\psi_\omega$. The above equation can be interpreted as a map from the state space $\mathcal{S}(\mathcal{E})$ of $\mathcal{E}$, to the polynomial algebra of the free Bose field. Of course, this map cannot local in that the set $\{\psi_\omega : \omega \in \mathcal{S}(\mathcal{E})\}$ does not contain a sequence approximating the *delta*. More informations on the way in which locality get broken can be obtained by computing the commutator

$$[\phi_m(\omega_a), \phi_m(\omega_b)] \subset \langle \psi_{\omega_1}, \Delta_m \psi_{\omega_2)} \rangle I,$$

where the distribution $\Delta_m$ is the usual propagator, and $\omega_a, \omega_b$ are states of optimal localization, with localization centers $a$ and $b$, respectively. We refer to the original paper for a discussion.

It was found in [13], that the free Hamiltonian $H = \int d^3\mathbf{x}\ \mathcal{H}(t, \mathbf{x})$ of the free field fulfills

$$H \otimes 1 = \int_{q_0 = t} d^3q\ \mathcal{H}(q),$$

where

$$\mathcal{H}(x) = \frac{1}{2} \left[ \left( \frac{\partial \phi_m}{\partial x_0} \right)^2 - \left( \frac{\partial^2 \phi_m}{\partial {x_0}^2}(x) \right) \phi_m(x) \right],$$

and 1 is the identity in $M(\mathcal{E})$.

In other words, the hamiltonian of the free field is the same on both the classical and quantum spacetime. Moreover, it was also found that

$$\left( \Box + m^2 \right) \phi_m(q) = 0,$$

where $\Box$ acts on $M(\mathcal{E})$, and is defined in terms of the partial derivatives on $M(\mathcal{E})$:

$$\partial_\mu f(q) = \frac{\partial}{\partial a_\mu} f(q + aI) \bigg|_{a=0}.$$

The following perturbative approach was proposed in [13]: since

$$\langle \phi_m(q), f(q) \rangle_t \equiv \int_\Sigma d\sigma \int_{q_0 = t} d^3q\ \phi_m(q) \overleftrightarrow{\partial}_0 f(q)$$

is a well defined quadratic form, independent on $t$ for any solution $f(q) \in M(\mathcal{E})$ of the Klein–Gordon equation, the interacting field is defined by

$$\langle \phi_{\mathrm{int}}(q), f(q) \rangle_t \equiv U(t,0) \langle \phi_m(q), f(q) \rangle U(0,t),$$

where $U(t,s)$ is, as usual, a solution of

$$\begin{gathered} U(t,s)U(s,r) = U(t,r), \\ U(t,t) = 1, \\ \frac{d}{dt} U(t,s) = iH_I(t) U(t,s). \end{gathered}$$

Using the map $f \mapsto f(q)$ from test functions to multipliers, we may interpret the field $\phi_{\mathrm{int}}$ as an equivalent field theory on the ordinary spacetime:

$$\langle \phi^{\mathrm{eq}}_{\mathrm{int}}, f \rangle_t = \langle \phi_{\mathrm{int}}(q), f(q) \rangle_t,$$

where

$$\langle \phi^{\mathrm{eq}}_{\mathrm{int}}, f \rangle_t \equiv \int d^3\mathbf{x} \, \big( \phi^{\mathrm{eq}}_{\mathrm{int}} \overleftrightarrow{\partial}_0 \, f \big)(t, \mathbf{x})$$

as usual.

The interaction hamiltonian proposed by[13] was

$$H_I(t) = \int_{\sigma^{(1)}} d\sigma \int d^3 q_{t=0} \colon \phi_m(q)^n \colon, \tag{1.1}$$

where

$$\colon \phi_m(q) \colon \equiv \int dk \, \colon \phi_m(k)^n \colon \otimes e^{ikq}.$$

Notice that $\int q_0 = t d^3 q : \mathcal{E} \to Z$; so that in general $\int_{q_0 = t} d^3 q \; f(q)$ depends on $\sigma$. To obtain a complex number, one still has to integrate over $\Sigma$, i.e. to evaluate a state on $Z$. Unfortunately, there is no obvious Lorentz invariant choice, since $L^{\uparrow}_{+}$ is not amenable. The most natural choice, then, is to consider the Lebesgue measure on the subset $\Sigma^{(1)}$ of $\Sigma$, the matrices in which correspond to the pure states with optimal localization (see [13]). This of course lead to a rotation but not Lorentz covariant definition.

There are indications of a partial removal of ultraviolet divergences in this class of models. A thorough investigation will be presented in [1]. Notice that the graphwise approach currently adopted in the literature (see nearly any paper referring to [17]) is not related to the approach presented here. See section 3.2 for more details